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# Rational solutions of a differential-difference KdV equation, the Toda equation and the discrete KdV equation

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**Abstract.** In this paper, a series of rational solutions are presented for a differential-difference analogue of the KdV equation, the Toda equation and the discrete KdV equation. These rational solutions are obtained using Hirota's bilinear formalism and Bäcklund transformations. The crucial step is the use of nonlinear superposition formulae.

## 1. Introduction

It is known that to find exact solutions of differential equations is always one of the central themes of perpetual interest in mathematics and physics. With the development of soliton theory, significant progress has been made in finding special solutions of integrable nonlinear evolution equations which include soliton solutions, rational solutions, similarity solutions and so on, and many approaches have been developed such as the inverse scattering transform (IST) [1, 2], Hirota's bilinear method [3], Bäcklund transformations [4], the Lie symmetry method [5, 6], the 'non-classical method' due to Bluman and Cole [7], the 'direct method' due to Clarkson and Kruskal [8].

In this paper we derive rational solutions of a differential-difference KdV equation, the Toda equation and the discrete KdV equation using Hirota's bilinear formalism and Bäcklund transformations. Here the crucial step in finding rational solutions of these equations is the use of nonlinear superposition formulae. We remark that many results have been obtained with respect to finding special solutions of $(1+1)$-dimensional and $(2+1)$-dimensional integrable nonlinear evolution equations. However, in comparison with the continuous case, there seems to be relatively less work for discrete integrable equations, though discrete Painlevé equations have attracted much recent attention, cf [9–16].

This paper is organized as follows. In section 2, a differential-difference KdV equation is considered and rational solutions of this equation are obtained. In section 3, we obtain the similar results for the Toda equation. In section 4, the rational solutions of a discrete KdV equation are given. Finally, conclusion and discussion are given in section 5.

## 2. The differential-difference KdV equation

The differential-difference KdV equation under consideration is [17]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{w_n}{1+w_n}\right) = w_{n-\frac{1}{2}} - w_{n+\frac{1}{2}}. \tag{1}$$

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By means of a variable transformation

$$w_n = \frac{\cosh(\frac{1}{2}D_n) f_n \bullet f_n}{f_n^2} - 1.$$

equation (1) is reduced to the bilinear equation [18]

$$\sinh(\tfrac{1}{4}D_n)\left[D_t + 2\sinh(\tfrac{1}{2}D_n)\right] f_n \bullet f_n = 0 \tag{2}$$

where the bilinear operators are defined as follows [3]:

$$D_x^m D_t^n a(x,t) \bullet b(x,t) \equiv (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n a(x,t) b(x',t')|_{x'=x,t'=t}$$

$$\exp(\delta D_n) a_n \bullet b_n \equiv \exp[\delta(\partial_n - \partial_{n'})] a(n) b(n')|_{n'=n} = a(n+\delta) b(n-\delta)$$

where $\partial_x \equiv \partial/\partial x$ etc. A Bäcklund transformation for (2) is given by Hirota [18]

$$\cosh(\tfrac{1}{2}D_n) f_n \bullet f_n' = \lambda f_n f_n' \qquad [D_t + 2\lambda \sinh(\tfrac{1}{2}D_n)] f_n \bullet f_n' = 0. \tag{3}$$

If we take $\lambda = 1$, then (3) becomes

$$\cosh(\tfrac{1}{2}D_n) f_n \bullet f_n' = f_n f_n' \qquad [D_t + 2\sinh(\tfrac{1}{2}D_n)] f_n \bullet f_n' = 0. \tag{4}$$

We shall represent the transformation (4) symbolically by $f_n \longrightarrow f_n'$. Henceforth, we denote $f_n(t) \equiv f(n,t) \equiv f(n) \equiv f$.

In this section, we obtain a hierarchy of rational solutions for (1). To do this, we first prove the following result.

*Proposition 1.* Let $f_0$ and $f_1$ be two solutions of (2) and $f_0 \longrightarrow f_1$, with $f_0 \neq 0$ and $f_1 \neq 0$. Then there exists a $f_2$ determined by

$$\sinh(\tfrac{1}{4}D_n) f_1 \bullet f_2 = \frac{1}{k}\cosh(\tfrac{1}{4}D_n) f_0 \bullet f_0 \tag{5}$$

where $k$ is a non-zero constant, such that $f_2$ is a new solution of (2) and $f_0 \longrightarrow f_2$.

*Proof.* First we choose a particular solution $F$ from (5), i.e.

$$\sinh(\tfrac{1}{4}D_n) f_1 \bullet F = \frac{1}{k}\cosh(\tfrac{1}{4}D_n) f_0 \bullet f_0. \tag{6}$$

We have, by using (6),

$$\begin{aligned}
0 = f_0[\cosh(\tfrac{1}{2}D_n) - 1] f_1 \bullet f_0 \\
&= \tfrac{1}{2} f_1(n+\tfrac{1}{2}) \exp(-\tfrac{1}{4}\partial_n)[\cosh(\tfrac{1}{4}D_n) f_0 \bullet f_0] \\
&\quad + \tfrac{1}{2} f_1(n-\tfrac{1}{2}) \exp(\tfrac{1}{4}\partial_n)[\cosh(\tfrac{1}{4}D_n) f_0 \bullet f_0] - f_1(n) f_0^2(n) \\
&= \tfrac{1}{2} f_1(n+\tfrac{1}{2}) k \exp(-\tfrac{1}{4}\partial_n)[\sinh(\tfrac{1}{4}D_n) f_1 \bullet F] \\
&\quad + \tfrac{1}{2} f_1(n-\tfrac{1}{2}) k \exp(\tfrac{1}{4}\partial_n)[\sinh(\tfrac{1}{4}D_n) f_1 \bullet F] - f_1(n) f_0^2(n) \\
&= f_1(n)[\tfrac{1}{2} k \sinh(\tfrac{1}{2}D_n) f_1 \bullet F - f_0^2(n)]
\end{aligned}$$

which implies that

$$\sinh(\tfrac{1}{2}D_n) f_1 \bullet F = \frac{2}{k} f_0^2. \tag{7}$$

Next, we have by using equations (6), (7) and (A1)–(A4):

$$\sinh(\tfrac{1}{4}D_n)\left[D_t f_1 \bullet F + \frac{4}{k}\cosh(\tfrac{1}{2}D_n) f_0 \bullet f_0\right] \bullet f_1^2$$

$$= D_t[\sinh(\tfrac{1}{4}D_n) f_1 \bullet F] \bullet [\cosh(\tfrac{1}{4}D_n) f_1 \bullet f_1]$$

$$+\frac{4}{k}\sinh(\tfrac{1}{4}D_n)[\cosh(\tfrac{1}{2}D_n)] f_0 \bullet f_0] \bullet f_1^2$$

$$= \frac{1}{k} D_t[\cosh(\tfrac{1}{4}D_n) f_0 \bullet f_0] \bullet [\cosh(\tfrac{1}{4}D_n) f_1 \bullet f_1]$$

$$+\frac{4}{k}\sinh(\tfrac{1}{4}D_n)[\cosh(\tfrac{1}{2}D_n)] f_0 \bullet f_0] \bullet f_1^2$$

$$= \frac{2}{k}\cosh(\tfrac{1}{4}D_n)(D_t f_0 \bullet f_1) \bullet f_0 f_1]$$

$$+\frac{4}{k}\sinh(\tfrac{1}{4}D_n)[\cosh(\tfrac{1}{2}D_n)] f_0 \bullet f_0] \bullet f_1^2$$

$$= -\frac{4}{k}\cosh(\tfrac{1}{4}D_n)[\sinh(\tfrac{1}{2}D_n) f_0 \bullet f_1] \bullet f_0 f_1$$

$$+\frac{4}{k}\sinh(\tfrac{1}{4}D_n)[\cosh(\tfrac{1}{2}D_n)] f_0 \bullet f_0] \bullet f_1^2$$

$$= \frac{4}{k}\sinh(\tfrac{1}{4}D_n)[\cosh(\tfrac{1}{2}D_n) f_0 \bullet f_1] \bullet f_0 f_1 = 0 \tag{8}$$

which implies that

$$D_t f_1 \bullet F + \frac{4}{k}\cosh(\tfrac{1}{2}D_n) f_0 \bullet f_0 = c(t) f_1^2 \tag{9}$$

where $c(t)$ is a suitable function of $t$. Now we choose $f_2 = F + f_1 \int^t c(t')\,dt'$. Then equation (8) becomes

$$D_t f_1 \bullet f_2 + \frac{4}{k}\cosh(\tfrac{1}{2}D_n) f_0 \bullet f_0 = 0 \tag{10}$$

while $f_2$ satisfies (5). Using equations (5) and (9), we can deduce that

$$[\cosh(\tfrac{1}{2}D_n) - 1] f_0 \bullet f_2 = 0 \qquad [D_t + 2\sinh(\tfrac{1}{2}D_n)] f_0 \bullet f_2 = 0. \tag{11}$$

Therefore $f_2$ is a solution of (2) and $f_0 \longrightarrow f_2$. Thus we have completed the proof of proposition 1.

As an application of proposition 1, we can obtain a hierarchy of polynomial solutions of (2). For example, if we choose $f_0 = n - t + A$ and $f_1 = 1$, with $A$ an arbitrary constant, then it is easily verified that $n - t + A$ and 1 are two solutions of (2) and $n - t + A \longrightarrow 1$. Furthermore, we can show that

$$f_2 = (n-t)^3 + 3A(n-t)^2 + (3A^2 - \tfrac{1}{4})(n-t) + \tfrac{1}{2}t + B$$

satisfies (5) with $k = -\frac{4}{3}$, where $B$ is an arbitrary constant. Further suppose

$$f_0 = (n-t)^3 + 3A(n-t)^2 + (3A^2 - \tfrac{1}{4})(n-t) + \tfrac{1}{2}t + B \qquad f_1 = n - t + A.$$

Then we seek a solution in the form

$$f_2=(n-t)^6+a_1(t)(n-t)^5+a_2(t)(n-t)^4+a_3(t)(n-t)^3 +a_4(t)(n-t)^2+a_5(t)(n-t)+a_6(t)$$

such that equations (5) and (9) hold. A direct calculation shows that

$$a_1=6A$$
$$a_2=-\tfrac{5}{4}+15A^2$$
$$a_3=\tfrac{5}{2}t+5B-\tfrac{15}{4}A+15A^3$$
$$a_4=\tfrac{1}{4}-\tfrac{15}{4}A^2+\tfrac{15}{2}At+15AB$$
$$a_5=(\tfrac{15}{2}A^2-1)t+C$$
$$a_6=-\tfrac{5}{4}t^2+(\tfrac{15}{2}A^3-\tfrac{9}{4}A-5B)t+\tfrac{15}{4}A^4-5B^2-\tfrac{5}{2}AB-\tfrac{9}{16}A^2+AC$$

where $C$ is an arbitrary constant. In general, suppose we have two solutions of (2) given by

$$F_{N(N+1)/2}=(n-t)^{N(N+1)/2}+[a_1(t)]^{N(N+1)/2}(n-t)^{N(N+1)/2-1} +\cdots+[a_{N(N+1)/2}(t)]^{N(N+1)/2}$$

$$F_{(N+1)(N+2)/2}=(n-t)^{(N+1)(N+2)/2}+[a_1(t)]^{(N+1)(N+2)/2}(n-t)^{(N+1)(N+2)/2-1} +\cdots+[a_{(N+1)(N+2)/2}(t)]^{(N+1)(N+2)/2}$$

such that $F_{N(N+1)/2}\longrightarrow F_{(N+1)(N+2)/2}$. Then, by seeking a solution in the form

$$F_{(N+2)(N+3)/2}=(n-t)^{(N+2)(N+3)/2}+[a_1(t)]^{(N+2)(N+3)/2}(n-t)^{(N+2)(N+3)/2-1} +\cdots+[a_{(N+2)(N+3)/2}(t)]^{(N+2)(N+3)/2}$$

such that

$$\sinh(\tfrac{1}{4}\mathrm{D}_n)F_{N(N+1)/2}\bullet F_{(N+2)(N+3)/2} = -\tfrac{1}{4}(2N+3)\cosh(\tfrac{1}{4}\mathrm{D}_n)F_{(N+1)(N+2)/2}\bullet F_{(N+1)(N+2)/2}$$

$$\mathrm{D}_tF_{N(N+1)/2}\bullet F_{(N+2)(N+3)/2}-(2N+3)\cosh(\tfrac{1}{2}\mathrm{D}_n)F_{(N+1)(N+2)/2}\bullet F_{(N+1)(N+2)/2}=0$$

hold, we can deduce a series of polynomial solutions of (3) and so from (2) we obtain the rational solutions of (1) as given in the following table.

| $f_n(t)$ | $w_n(t)$ |
|---|---|
| $n-t+A$ | $-\dfrac{1}{4(n-t+A)^2}$ |
| $(n-t)^3+3A(n-t)^2+(3A^2-\frac{1}{4})(n-t)+\frac{1}{2}t+B$ | $\dfrac{F}{G}$ |

where

$$F=-\tfrac{3}{4}(n-t)^4-3A(n-t)^3+(\tfrac{3}{16}-\tfrac{9}{2}A^2)(n-t)^2+(\tfrac{3}{4}t+\tfrac{3}{2}B-\tfrac{9}{2}A^3+\tfrac{3}{4}A)(n-t) +\tfrac{3}{4}At+\tfrac{3}{2}AB+\tfrac{9}{16}A^2-\tfrac{9}{4}A^4$$

and

$$G=[(n-t)^3+3A(n-t)^2+(3A^2-\tfrac{1}{4})(n-t)+\tfrac{1}{2}t+B]^2.$$

## 3. The Toda equation

The Toda equation is [19]

$$\frac{\partial^2}{\partial t^2}\ln[1+V_n(t)] = V_{n+1}(t)+V_{n-1}(t)-2V_n(t). \tag{12}$$

By a variable transformation

$$V_n(t) = \frac{\partial^2}{\partial t^2}\ln f_n(t) \tag{13}$$

equation (12) is transformed into the bilinear equation [3]

$$[\mathrm{D}_t^2 - 4\sinh^2(\tfrac{1}{2}\mathrm{D}_n)]f \bullet f = 0. \tag{14}$$

A Bäcklund transformation for (13) is given by Hirota and Satsuma [20]:

$$\cosh(\tfrac{1}{2}\mathrm{D}_n)f_n \bullet f_n' = \lambda f_n f_n' \qquad [\mathrm{D}_t + 2\lambda\sinh(\tfrac{1}{2}\mathrm{D}_n)]f_n \bullet f_n' = 0. \tag{15}$$

Since the Bäcklund transformation (15) of the Toda equation (14) is the same as that for the differential-difference KdV equation, i.e. (3), we can also deduce a series of rational solutions of (14), and so from (15) we obtain the rational solutions of the Toda equation (12) as given in the following table.

| $f_n(t)$ | $V_n(t)$ |
|---|---|
| $n-t+A$ | $-\dfrac{1}{(n-t+A)^2}$ |
| $(n-t)^3+3A(n-t)^2+(3A^2-\frac{1}{4})(n-t)+\frac{1}{2}t+B$ | $\dfrac{F}{G}$ |

where

$$F = -3(n-t)^4 - 12A(n-t)^3 + (3-18A^2)(n-t)^2 + (3t+6B-18A^3+\tfrac{15}{2}A)(n-t) \\ +3At+6AB+\tfrac{9}{2}A^2-9A^4-\tfrac{9}{16}$$

and

$$G = [(n-t)^3+3A(n-t)^2+(3A^2-\tfrac{1}{4})(n-t)+\tfrac{1}{2}t+B]^2.$$

## 4. The discrete analogue of the KdV equation

The so-called discrete analogue of the KdV equation is [3, 18]

$$\Delta_t\frac{W_n(t)}{1+W_n(t)} = W_{n-1/2}(t) - W_{n+1/2}(t) \tag{16}$$

where $\Delta_t$ is a central difference operator defined by

$$\Delta_t F(t) = \delta^{-1}[F(t+\delta/2) - F(t-\delta/2)].$$

By means of the variable transformation

$$W_n(t) = \frac{f_{n+1/2}(t)f_{n-1/2}(t)}{f_n(t+\delta/2)f_n(t-\delta/2)} - 1 \tag{17}$$

equation (16) is transformed into the bilinear equation [3, 17]

$$\sinh\tfrac{1}{4}(\mathrm{D}_n+\delta\mathrm{D}_t)[2\delta^{-1}\sinh(\tfrac{1}{2}\delta\mathrm{D}_t) + 2\sinh(\tfrac{1}{2}\mathrm{D}_n)]f \bullet f = 0. \tag{18}$$

A Bäcklund transformation for (18) is given by Hirota [3, 18]

$$\cosh(\tfrac{1}{2}D_n)f\bullet f' = \lambda\cosh(\tfrac{1}{2}\delta D_t)f\bullet f' \qquad [2\delta^{-1}\sinh(\tfrac{1}{2}\delta D_t) + 2\lambda\sinh(\tfrac{1}{2}D_n)]f\bullet f' = 0. \tag{19}$$

We take $\lambda = 1$, then (19) becomes

$$\cosh(\tfrac{1}{2}D_n)f\bullet f' = \cosh(\tfrac{1}{2}\delta D_t)f\bullet f' \qquad [2\delta^{-1}\sinh(\tfrac{1}{2}\delta D_t) + 2\sinh(\tfrac{1}{2}D_n)]f\bullet f' = 0. \tag{20}$$

We shall represent (20) symbolically by $f \longrightarrow f'$. We have the following result.

*Proposition 2.* Suppose $f_0$ and $f_1$ are two solutions of (18) and $f_0 \longrightarrow f_1$ such that $f_0 \neq 0$ and $f_1 \neq 0$. Then $f_2$ given by

$$\sinh\tfrac{1}{4}(D_n + \delta D_t)f_1\bullet f_2 = \frac{1}{k}\cosh\tfrac{1}{4}(D_n + \delta D_t)f_0\bullet f_0 \tag{21}$$

$$\sinh\tfrac{1}{4}(D_n - \delta D_t)f_1\bullet f_2 = \frac{1}{k}\frac{1+\delta}{1-\delta}\cosh\tfrac{1}{4}(D_n - \delta D_t)f_0\bullet f_0 \tag{22}$$

where $k$ is a non-zero constant, is a new solution of (18) and $f_0 \longrightarrow f_2$.

The proof of this proposition is similar to that for proposition 1 and so details are left to the reader.

Using proposition 2, we can also obtain a series of polynomial solutions of (9). For example, if choose $f_0 = n - t + A$, $f_1 = 1$, where $A$ is an arbitrary constant, we then obtain

$$f_2 = (n-t)^3 + 3A(n-t)^2 + (3A^2 - \tfrac{3}{4}\delta^2 - \tfrac{1}{4})(n-t) + \tfrac{1}{2}(1-\delta^2)t + B$$

with $B$ an arbitrary constant, which satisfies (21) and (22) with $k^{-1} = \frac{3}{4}(\delta - 1)$. Thus from (20) we obtain rational solutions of (19) as given in the following table.

| $f_n(t)$ | $W_n(f_n(t))$ |
|---|---|
| $n - t + A$ | $\dfrac{\delta^2 - 1}{4(n-t+A)^2 - \delta^2}$ |
| $(n-t)^3 + 3A(n-t)^2 + (3A^2 - \frac{3}{4}\delta^2 - \frac{1}{4})(n-t)$ $+\frac{1}{2}(1-\delta^2)t + B$ | $\dfrac{G_1}{G_2} - 1$ |

where

$$G_1 = \big[(n-t)^3 + \tfrac{3}{4}(n-t) + 3A(n-t)^3 + \tfrac{3}{4}A + (3A^2 - \tfrac{3}{4}\delta^2 - \tfrac{1}{4})(n-t)$$
$$+\tfrac{1}{2}(1-\delta^2)t + B\big]^2 - \big[\tfrac{3}{2}(n-t)^2 + \tfrac{1}{8} + 3A(n-t)$$
$$+\tfrac{1}{2}(3A^2 - \tfrac{3}{4}\delta^2 - \tfrac{1}{4})\big]^2 + \tfrac{9}{16}A^2 - \tfrac{9}{4}A^4$$

and

$$G_2 = [(n-t)^3 + \tfrac{3}{4}\delta^2(n-t) + \tfrac{3}{4}A\delta^2 + 3A(n-t)^2 + (3A^2 - \tfrac{3}{4}\delta^2 - \tfrac{1}{4})(n-t)$$
$$+\tfrac{1}{2}(1-\delta^2)t + B]^2 - [\tfrac{3}{2}\delta(n-t)^2 + \tfrac{1}{8}\delta^3 + 3A\delta(n-t)$$
$$+\tfrac{1}{2}\delta(3A^2 - \tfrac{3}{4}\delta^2 - \tfrac{1}{4})]^2.$$

## 5. Discussion and conclusion

In this paper, we have obtained rational solutions of the differential-difference KdV equation, Toda equation and the discrete KdV equation. There are many methods to obtain rational solutions. The method used here is Hirota's bilinear formalism and Bäcklund transformations. The crucial step is the use of corresponding nonlinear superposition formulae. It is of interest to note that the rational solutions thus obtained for the differential-difference KdV equation, Toda equation and the discrete KdV equation are connected by Bäcklund transformations which are special cases of more general Bäcklund transformations with parameters. Thus it enables us to obtain other new solutions of these equations through nonlinear superposition of soliton solutions and rational solutions. The simplest example was given in [21] for the differential-difference KdV equation. To be more precise, take one-soliton solution and rational solution of (2), i.e. $\exp(\eta)+\exp(-\eta)$ and $t-n$, respectively. The new solution of (2) generated through nonlinear superposition of these two solutions is

$$2(t-n)\sinh(\tfrac{1}{2}p)\sinh(\eta)+\cosh(\tfrac{1}{2}p)\cosh(\eta)$$

where $\eta=\sinh(p)t+\eta_0$, with $p$ and $\eta_0$ as constants. Thus using the results in this paper and [21, 22], we can obtain further solutions of differential-difference KdV equation (1), the Toda equation (12) and the discrete KdV equation (18).

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## Appendix

The following bilinear operator identities hold for arbitrary functions $a$ and $b$ :

$$\sinh(\tfrac{1}{4}D_n)(D_t a\bullet b)\bullet a^2 = D_t[\sinh(\tfrac{1}{4}D_n)a\bullet b]\bullet[\cosh(\tfrac{1}{4}D_n)a\bullet a] \tag{A1}$$

$$D_t[\cosh(\tfrac{1}{4}D_n)a\bullet a]\bullet[\cosh(\tfrac{1}{4}D_n)b\bullet b] = 2\cosh(\tfrac{1}{4}D_n)(D_t a\bullet b)\bullet ab \tag{A2}$$

$$\begin{aligned}\cosh(\tfrac{1}{4}D_n)[\sinh(\tfrac{1}{2}D_n)a\bullet b]\bullet ab - \sinh(\tfrac{1}{4}D_n)[\cosh(\tfrac{1}{2}D_n)a\bullet a]\bullet b^2\\ = -\sinh(\tfrac{1}{4}D_n)[\cosh(\tfrac{1}{2}D_n)a\bullet b]\bullet ab\end{aligned} \tag{A3}$$

$$\sinh(\tfrac{1}{4}D_n)a\bullet a = 0. \tag{A4}$$

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